Math 255A' Lecture 24 Notes

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1 Banach Algebras

1.1 Convolution of measures

Understanding Banach algebras will help us obtain a better understanding of the spectral theorem.

Here is a motivating example.

Example 1.1. Let G be a locally compact Hausdorff group (so you can think of $G = \mathbb{R}^d$). Then there is a space $M(G)$ of finite, regular, $\mathbb{F}$-valued Borel measures. Given $\mu, \nu \in M(G)$, define the **convolution**

$$\mu * \nu(A) := \mu \times (\{(g,h) : gh \in A\}).$$

This is related to convolution of functions: $d\mu = f\,dm$ and $d\nu = f'\,dm$, then $f(\mu * \nu) = (f * f')\,dm$. We could alternatively define this by its action on $f \in C_0(G)$:

$$\int f\,d(\mu * \nu) = \iint f(gh)\,d\mu(g)\,d\nu(h).$$

This is distributive over addition, and associative:

$$\int f\,d((\mu * \nu) * \lambda) = \iint f(ghk)\,f\mu(g)\,d\nu(h)\,d\lambda(k).$$

Observe that

$$\left|\int f\,d(\mu * \nu)\right| = \left|\iint f(gh)\,d\mu(g)\,d\nu(h)\right| \leq \iint |f(gh)|\,d|\mu|\,d|\nu| \leq \|f\| \cdot \|\mu\| \cdot \|\nu\|,$$

so $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$.

There is also an identity element with respect to convolution, δ_e. We have

$$(\delta_e * \mu)(A) = (\delta_e \times \mu)(\{(g,h) : gh \in A\}) = \mu(\{h : h \in A\}) = \mu(A),$$

and a similar property holds for right multiplication by δ_e. You can also check that $\delta_g * \delta_h = \delta_{gh}$. So $M(X)$ is a unital Banach algebra with convolution as the multiplication.

1.2 Invertibility and ideals

Definition 1.1. Let $\mathscr{A}$ be a Banch algebra. Then $x \in \mathscr{A}$ is **left-invertible** if there is some $y \in \mathscr{A}$ such that $yx = 1$, **right-invertible** if there is some $y \in \mathscr{A}$ such that $xy = 1$, and **invertible** if it is left and right invertible.

If x is left and right invertible, the inverses are the same: $z = yxz = y$. We write this as x^{-1}.

One important question is: Given an algebra, can we recover information about what generated it?

Definition 1.2. M is a **left ideal** in $\mathscr{A}$ if M is a vector subspace and $xy \in M$ for all $x \in A$ and $y \in M$. M is a **right ideal** in $\mathscr{A}$ if $\mathscr{M}$ is a vector subspace and $yx \in M$ for all $x \in A$ and $y \in M$. M is an **ideal** if it is a left and right ideal.

Example 1.2. The compact operators, $\mathcal{B}_0(X) \subseteq \mathcal{B}(X)$, form an ideal.

Example 1.3. Let $X \neq \varnothing$ be compact, and let $K \subsetneq X$ be closed and nonempty. Then $C(X) \supseteq \{f \in C(X) : f|_K = 0\} =: I(K)$. Then $K \subseteq L \iff I(L) \subseteq I(K)$.

These get bigger if K gets smaller In fact, there is a correspondence between maximal ideals of $C(X)$ and points of X. So we can recover X from $C(X)$.

Lemma 1.1. *Let $\mathscr{A}$ is a Banach algebra with identity, and let $x \in \mathscr{A}$ have $\|x - 1\| < 1$. Then x is invertible.*

Proof. Let $y := \sum_{k=0}^{\infty}(1-x)^k$. The norm of the k-th term is $\leq 1\|1-x\|^k$. So this is an absolutely convergent series. So for any $z \in \mathcal{A}$, we have $zy = \sum_{k=0}^{\infty} z(1-x)^k$. This gives

$$(1-x)y = \sum_{k=0}^{\infty}(1-x)(1-x)^k = \sum_{k=0}^{\infty}(1-x)^{k+1} = y - (1-x)^0.$$

So we get $xy = (1-x)^0 = 1$. $\square$

Corollary 1.1. *If $\|x - 1\| < \varepsilon < 1$, then $\|x^{-1} - 1\| < \frac{\varepsilon}{1-\varepsilon}$.*

Proof.

$$\|x^{-1} - 1\| = \left\| \sum_{k=1}^{\infty}(1-x)^k \right\| \leq \sum_{k=1}^{\infty} \|1-x\|^k < \frac{\varepsilon}{1-\varepsilon}.$$ $\square$

Corollary 1.2. *If $ba = 1$ and $\|c - a\| < 1/\|b\|$, then c is left-invertible.*

Proof. We have

$$\|bc - 1\| = \|bc - ba\| \leq \|b\|\|c - a\| < 1.$$

so there is an $x = (bc)^{-1}$. So $(xb)c = 1$ means that xb is the inverse of c. $\square$

Proposition 1.1. *Let $\mathscr{A}$ be a Banach algebra with identity, let G_ℓ be the left-invertible elements, let G_r be the right-invertible elements, and let $G = G_\ell \cap G_r$. Moreover, the map $G \to G : x \mapsto x^{-1}$ is continuous.*

Proof. Openness follows from the previous corollary. For continuity, if $x \in G$, suppose that $\|y - x\| < \varepsilon^{-1}$ for some small enough $\varepsilon > 0$. Then $\|x^{-1}y - 1\| < \varepsilon\|x^{-1}\| < 1$. So

$$\|(x^{-1}y)^{-1} - 1\| < \frac{\varepsilon\|x^{-1}\|}{1 - \varepsilon\|x^{-1}\|}.$$

Then y^{-1} exists (because it is equal to $(x^{-1}y)^{-1}x^{-1}$, and

$$\|y^{-1} - x^{-1}\| < \frac{\varepsilon\|x^{-1}\|^2}{1 - \varepsilon\|x^{-1}\|}.$$ □

1.3 Maximal ideals and quotients

Definition 1.3. A left/right/two-sided ideal M is **maximal** if it is

1. proper ($M \neq A$),
2. M is not properly contained in any other proper ideal.

Corollary 1.3. *If $\mathscr{A}$ has an identity, then*

1. *The closed of a left/right/two-sided ideal is an ideal of the same kind.*
2. *Maximal ideals are closed.*

Proof. Check the proof of (1).

If M is a maximal (e.g. two-sided) ideal, then $M \cap G_\ell = \varnothing$. This is because if $x \in M \cap G_\ell$, then there exists some y such that $yx = 1$. So $1 \in M$, but then $a = a1 \in M$ for all $a \in \mathscr{A}$. So $M = \mathscr{A}$. In fact, we have $\overline{M} \cap G_\ell = \varnothing$. Now $M = \overline{M}$ by maximality. □

Example 1.4. The algebra $C_0(\mathbb{R}) \supseteq C_c(\mathbb{R}) = \{f : f|_{[-a,a]^c} = 0 \text{ for some } a\}$. This is a dense ideal. This tells us that this fact really relies on the existence of an identity.

Proposition 1.2. *Any proper (left/right/two-sided) ideal in any algebra is contained in a maximal (left/right/two-sided) ideal.*

Proof. Zorn's lemma. □

Lemma 1.2. *Let $\mathscr{A}$ be a Banach algebra, and let M be a closed idea in $\mathscr{A}$. Then $\mathscr{A}/M$ is still a Banach algebra.*

Proof. Given $(x+M),(y+M) \in \mathscr{A}/M$, define $(x+M)(y+M) := xy+M$. To show that this is well-defined, we have that for any $m,n \in M$,

$$(x+m+M)(y+n+M) = xy + \underbrace{my + xn + mn}_{\in M} + M = xy + M.$$

To check that $\mathscr{A}/M$ is a Banach algebra, we have

$$\|(x+M)(y+M)\| = \|xy+M\| \leq \|xy\| \leq \|x\|\|y\|.$$

This is true for all x,y, so we can take the inf over x and y to get $\|(x+M)(y+M)\| \leq \|(x+M)\|\|(y+M)\|$. $\square$

1.4 The spectrum of an element

Definition 1.4. Let $\mathscr{A}$ have an identity, and let $x \in A$. The **spectrum** is $\sigma(x) = \{\lambda \in \mathbb{F} : x - \lambda$ not invertible, the **left-spectrum** is $\sigma_\ell(x) = \{\lambda \in \mathbb{F} : x - \lambda$ not left-invertible, and **right-spectrum** is $\sigma_r(x) = \{\lambda \in \mathbb{F} : x - \lambda$ not right-invertible. The **resolvent** is $\rho(x) = \mathbb{F} \setminus \sigma(x)$.

Example 1.5. Let X be a compact, Hausdorff space, and let $f \in C(X)$. Then $\sigma(f) = f(X)$ is the image of f. If $g(f-\lambda) = 1$, then $g(x) = \frac{1}{f(x)-\lambda}$ for all x.